Differential Geometry and its Applications 29 (2011) 597–605

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Sufficient conditions for open manifolds to be diffeomorphic to Euclidean spaces

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ARTICLE INFO

Article history:
Received 10 May 2010
Received in revised form 29 March 2011
Available online 22 April 2011
Communicated by J. Slovák

MSC:
primary 53C21
secondary 53C22

Keywords:
Volume growth
Radial curvature
Ricci curvature

ABSTRACT

Let M be a complete non-compact connected Riemannian n-dimensional manifold. We first prove that, for any fixed point $p \in M$, the radial Ricci curvature of M at p is bounded from below by the radial curvature function of some non-compact n-dimensional model. Moreover, we then prove, without the pointed Gromov–Hausdorff convergence theory, that, if *model* volume growth is sufficiently close to 1, then M is diffeomorphic to Euclidean n-dimensional space. Hence, our main theorem has various advantages of the Cheeger–Colding diffeomorphism theorem via the *Euclidean* volume growth. Our main theorem also contains a result of do Carmo and Changyu as a special case.

1. Introduction

In the geodesic theory of global Riemannian geometry, the critical point theory of distance functions, introduced by Grove and Shiohama [5], provides a useful application to study the relationship between the topology and geometry of a given Riemannian manifold. Here we say that a point q in a complete Riemannian manifold M is a *critical point of the distance function* $d(p,\cdot)$ to $p \in M$ (or a *critical point* q *for* p), if for every nonzero tangent vector v in the tangent space T_qM to q, there exists a minimal geodesic segment γ emanating from q to p satisfying $\angle(v,\gamma'(0)) \leqslant \pi/2$, where $\angle(v,\gamma'(0))$ denotes the angle between two vectors v and $\gamma'(0)$ in T_qM.

For complete non-compact Riemannian manifolds with bounded sectional curvature, this critical point theory becomes particularly useful when used in conjunction with Toponogov's comparison theorem. It is possible to investigate whether M has critical points or not by using the technique of drawing a circle or a geodesic polygon, joining two points by a minimal geodesic segment, and finally estimating the angles of geodesic triangles on M. If M admits a region which has no critical points, then the shape of the region can be stretched and deformed into a region on a plane (cf. [5], Corollary 1.4 in [10, Chapter 11]). In particular M is diffeomorphic to Euclidean n-dimensional space $\mathbb{R}^n$, if M does not have any critical points of $d(p,\cdot)$ for a fixed point $p \in M$.

To control a set of critical points of the distance function on a non-compact Riemannian n-dimensional manifold M with *non-negative* Ricci curvature *everywhere*, Otsu [9] very first introduced the Euclidean volume growth

$$\lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(x)}{t^n \,\mathrm{vol}\, \mathbb{S}^{n-1}(1)}, \tag{1.1}$$

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2011.04.040

where $\mathrm{vol}\, B_t(x)$ denotes the volume of the open distance ball $B_t(x)$ at a point $x \in M$ with radius $t > 0$ in M, and $\mathrm{vol}\, \mathbb{S}^{n-1}(1)$ denotes the volume of the unit ball $\mathbb{S}^{n-1}(1)$ in Euclidean n-dimensional space $\mathbb{R}^n$. Notice that, by the Bishop volume comparison theorem,

$$\lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(x)}{t^n \,\mathrm{vol}\, \mathbb{S}^{n-1}(1)} \leqslant 1.$$

If (1.1) equals 1, the M is isometric to $\mathbb{R}^n$. Hence, it is very natural to expect M to be diffeomorphic to $\mathbb{R}^n$, when (1.1) is sufficiently close to 1. In fact, Otsu proved

Theorem 1.1. *(See [9, Theorem 1.2].) Let M be a complete non-compact Riemannian n-manifold with non-negative Ricci curvature, and let $\lambda : [0, \infty) \to \mathbb{R}$ be a negative increasing continuous function such that*

(O-1) $c_0 := \int_0^\infty t\lambda(t)\, dt > -\infty$ *and that*
(O-2) *the sectional curvature at any point $q \in M$ is bounded from below by $\lambda(d(p, q))$ for some fixed point $p \in M$.*

Then, there exists $\delta(n, c_0) > 0$ such that, if

$$\lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(x)}{t^n \,\mathrm{vol}\, \mathbb{S}^{n-1}(1)} \geqslant 1 - \delta(n, c_0)$$

for some $x \in M$, then M is diffeomorphic to Euclidean n-space $\mathbb{R}^n$.

Notice that (O-1) and (O-2) imply that the manifold M is at least as curved as a model surface of revolution with a finite total curvature.

There is a *great* number of related results for Theorem 1.1. However, after Colding's study of the relationship between Ricci curvatures on complete Riemannian manifolds, Gromov–Hausdorff convergence theory and volumes of the manifolds [4], Cheeger and Colding proved the next theorem, which shines out very much among such related results:

Theorem 1.2. *(See [2, Theorem A.1.11].) Let M be a complete non-compact Riemannian n-manifold with non-negative Ricci curvature. Then, there exists $\delta(n) > 0$ such that, if*

$$\mathrm{vol}\, B_t(x) \geqslant \bigl(1 - \delta(n)\bigr) \mathrm{vol}\, \mathbb{S}^{n-1}(1) t^n$$

for all $x \in M$, $t > 0$, then M is diffeomorphic to Euclidean n-space $\mathbb{R}^n$.

Our purpose of this article is to extend Theorem 1.2 to any complete non-compact connected Riemannian manifold M, i.e., we will remove the non-negative Ricci curvature condition in Theorem 1.2. To state that precisely, we will begin by defining the radial curvature geometry.

Let $\widetilde{M}^n$ denote a complete non-compact Riemannian n-dimensional manifold, which is homeomorphic to $\mathbb{R}^n$, with a base point $\tilde{p} \in \widetilde{M}^n$. Then, we call the pair $(\widetilde{M}^n, \tilde{p})$ an n-dimensional *model* if its Riemannian metric $d\tilde{s}^2$ is expressed in terms of geodesic polar coordinates around $\tilde{p}$ as

$$d\tilde{s}^2 = dt^2 + f(t)^2\, d\theta^2, \quad (t, \theta) \in (0, \infty) \times \mathbb{S}^{n-1}_{\tilde{p}}. \tag{1.2}$$

Here $f : (0, \infty) \to \mathbb{R}$ is a positive smooth function which is extendible to a smooth odd function around 0, and $d\theta$ denotes the Riemannian metric on the unit sphere $\mathbb{S}^{n-1}_{\tilde{p}} := \{v \in T_{\tilde{p}}\widetilde{M}^n \mid \|v\| = 1\}$. The function $G \circ \tilde{\gamma} : [0, \infty) \to \mathbb{R}$ is called the *radial curvature function* of $(\widetilde{M}^n, \tilde{p})$, where we denote by G the sectional curvature of $\widetilde{M}^n$, and by $\tilde{\gamma}$ any meridian emanating from $\tilde{p} = \tilde{\gamma}(0)$. Note that f satisfies the differential equation

$$f''(t) + G\bigl(\tilde{\gamma}(t)\bigr) f(t) = 0$$

with initial conditions $f(0) = 0$ and $f'(0) = 1$. The n-models are completely classified in [6]. In particular, if $n = 2$, a model are called a *non-compact model surface of revolution*.

Let (M, p) be a complete non-compact Riemannian n-dimensional manifold with a base point $p \in M$. We say that (M, p) has *radial Ricci curvature at p bounded from below by the radial curvature function of an n-model* $(\widetilde{M}^n, \tilde{p})$ if, along every unit speed minimal geodesic $\gamma : [0, a) \to M$ emanating from $\gamma(0) = p$, its Ricci curvature Ric_p with respect to $\gamma'(t)$ satisfies

$$\mathrm{Ric}_p\bigl(\gamma'(t)\bigr) := \frac{1}{n-1} \sum_{i=1}^{n-1} \bigl\langle R\bigl(\gamma'(t), e_i\bigr)\gamma'(t), e_i\bigr\rangle \geqslant G\bigl(\tilde{\gamma}(t)\bigr)$$

for all $t \in [0, a)$. Here R denotes the Riemannian curvature tensor of M, which is a multi-linear map, defined by $R(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$ for smooth vector fields X, Y, Z over M and $\{e_1, e_2, \ldots, e_{n-1}\} := \{e_1(t), e_2(t), \ldots, e_{n-1}(t)\}$ denotes an orthonormal basis of the hyperplane in $T_{\gamma(t)}M$ orthogonal to $\gamma'(t)$. For example, if the Riemannian metric of $\widetilde{M}$

is $dt^2 + t^2\,d\theta^2$, or $dt^2 + \sinh^2 t\,d\theta^2$, then $G(\tilde\gamma(t)) = 0$, or $G(\tilde\gamma(t)) = -1$, respectively. Notice that *the radial Ricci curvature may change signs wildly*. For example, there exist model surfaces of revolution with finite total curvature whose Gauss curvatures are not bounded, i.e., such surfaces satisfy $\liminf_{t\to\infty} G(\tilde\gamma(t)) = -\infty$, or $\limsup_{t\to\infty} G(\tilde\gamma(t)) = \infty$ (see [13, Theorems 1.3 and 4.1]).

To state our main theorem, we need to introduce an essential ratio and an important function: Let (M, p) be a complete non-compact connected Riemannian n-dimensional manifold M whose radial Ricci curvature at the base point p is bounded from below by the radial curvature function of an n-dimensional model $(\widetilde M^n, \tilde p)$. Under this curvature relationship between M and $\widetilde M^n$, the limit

$$\lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t(\tilde p)}$$

is called the *model volume growth*, where $B_t(p) \subset M$ denotes the open distance ball at p with radius $t > 0$, and $B_t(\tilde p) \subset \widetilde M^n$ denotes the open distance ball at $\tilde p$ with radius $t > 0$. Furthermore we define a function

$$F(r) := \left(\int_0^\pi \sin^{n-2} t\,dt\right)^{-1} \int_0^r \sin^{n-2} t\,dt \tag{1.3}$$

on $[0, \pi]$, and we call it *the net function for* $\mathbb{S}_p^{n-1} := \{v \in T_pM \mid \|v\| = 1\}$.

Now our main theorem is stated as follows, which has various advantages of the Cheeger–Colding theorem above:

Main Theorem. *Let M be a complete non-compact connected Riemannian n-manifold. Then, for any fixed point $p \in M$,*

(A-1) *There exists a locally Lipschitz function $G(t)$ (respectively $K(t)$) on $[0, \infty)$ such that radial Ricci (respectively sectional) curvature of M at p is bounded from below by that of an n-model $(\widetilde M^n, \tilde p)$ with G (respectively that of a non-compact model surface of revolution with K) as its radial curvature function.*

(A-2) *Moreover, if*

(B-1) $\lim_{t\to\infty} \operatorname{vol} B_t(\tilde p) = \infty$ *and*

(B-2) $\lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t(\tilde p)} \geqslant 1 - F(\delta(K^*))$,

then M is diffeomorphic to Euclidean n-space $\mathbb{R}^n$. Here $B_t(p) \subset M$ and $B_t(\tilde p) \subset \widetilde M^n$ denote the open distance balls at p and at $\tilde p$ with radius $t > 0$, respectively, and we set $K^* := \min\{0, G, K\}$ and

$$\delta(K^*) := \frac{\pi}{2} \exp\left(\int_0^\infty tK^*(t)\,dt\right).$$

Here, we say that M has *radial sectional curvature at the base point $p \in M$ bounded from below by that of a non-compact model surface of revolution* if, along every unit speed minimal geodesic $\gamma : [0, a) \to M$ emanating from $p = \gamma(0)$, its sectional curvature $K_M(\sigma_t)$ is bounded from below by the radial curvature function of the surface for all $t \in [0, a)$ and all 2-dimensional linear spaces σ_t spanned by $\gamma'(t)$ and a tangent vector to M at $\gamma(t)$.

The first assertion (A-1) is already proved for the radial sectional curvature of the manifold at any fixed point (see [8, Lemma 5.1]).

In the second assertion (A-2), it is not necessary, as a condition, whether the value $\int_0^\infty tK^*(t)\,dt$ is finite or not. Moreover, the (A-2) has at least two advantages of Theorem 1.2, which are as follows:

(1) The condition (B-1) is natural, because we may easily find such a $(\widetilde M^n, \tilde p)$. For example, $\widetilde M^n = \mathbb{R}^n$.
(2) Our volume growth is bounded from below by a *definite* constant.

We may remove the condition (B-1) by assuming the radial sectional curvature is bounded below:

Corollary to Main Theorem. *Let (M, p) be a complete non-compact Riemannian n-manifold M whose radial sectional curvature at the base point p is bounded from below by the radial curvature function G of a non-compact model surface of revolution $(\widetilde M, \tilde p)$. If*

$$\lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t^n(\tilde p)} \geqslant 1 - F\big(\delta(G_-)\big)$$

then M is diffeomorphic to Euclidean n-space $\mathbb{R}^n$. Here $B_t^n(\tilde p)$ denotes the open distance ball at $\tilde p \in \widetilde M^n$ with radius $t > 0$ in an n-dimensional model $(\widetilde M^n, \tilde p)$, and we set $G_- := \min\{0, G\}$.

Hence, by our Main Theorem and this corollary, we realize that the difference between Ricci curvature and sectional curvature is whether the volume of each comparison model is finite or not. Notice that this corollary directly contains a results of do Carmo and Changyu [1] as a special case, that is, $f(t) = t$, where f is the warping function of the surface $(\widetilde{M}, \tilde{p})$.

In the following sections, all geodesics will be normalized, unless otherwise stated.

2. Mass of rays and volume growth

The purpose of this section is to investigate the relationship between mass of rays and the model volume growth. Especially, Lemma 2.3 is the key lemma to prove our main theorem. Since this lemma was stated in [9] without a proof, we will give a proof of it here.

Throughout this section, let (M, p) denote a complete non-compact Riemannian n-dimensional manifold M whose radial Ricci curvature at the base point p is bounded from below by the radial curvature function $G(\tilde{\gamma}(t))$ of an n-dimensional model $(\widetilde{M}^n, \tilde{p})$ with its metric (1.2). Let A_p be the set of all unit vectors tangent to rays emanating from $p \in M$. Then, it is clear that $A_p = \{v \in \mathbb{S}_p^{n-1} \mid \rho(v) = \infty\}$. Here we set

$$\rho(v) := \sup\bigl\{t > 0 \bigm| d\bigl(p, \gamma_v(t)\bigr) = t\bigr\},$$

where γ_v denotes the unit speed geodesic emanating from $p \in M$ such that $v = \gamma_v'(0) \in \mathbb{S}_p^{n-1}$. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of T_pM such that $e_n := v \in \mathbb{S}_p^{n-1}$. Take Jacobi fields $Y_i(t, v)$ along the unit speed geodesic γ_v emanating from $p \in M$ such that

$$Y_i(0, v) = 0, \qquad Y_i'(0, v) = e_i, \quad i = 1, 2, \dots, n-1.$$

Here Y_i' denotes the covariant derivative of Y_i along γ_v. Then, we set

$$\Theta(t, v) := \sqrt{\det\bigl(\bigl\langle Y_i(t, v), Y_j(t, v)\bigr\rangle\bigr)}, \quad 1 \leqslant i, j \leqslant n-1.$$

We define

$$\overline{\Theta}(t, v) = \begin{cases} \Theta(t, v), & t \leqslant \rho(v), \\ 0, & t > \rho(v). \end{cases}$$

Then,

$$\operatorname{vol} B_t(p) = \int_0^t dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r, v)\, d\mathbb{S}_p^{n-1}.$$

As well as above, for $(\widetilde{M}^n, \tilde{p})$, we may consider the corresponding notions $\mathbb{S}_{\tilde{p}}^{n-1}$, $\tilde{\gamma}_{\tilde{v}}$, $\widetilde{Y}_i(t, \tilde{v})$, $\widetilde{\Theta}(t, \tilde{v})$, etc. Since $\widetilde{\Theta}(t, \tilde{v}) = f(t)^{n-1}$, we have

$$\operatorname{vol} B_t(\tilde{p}) = \omega_{n-1} \int_0^t f(r)^{n-1}\, dr, \tag{2.1}$$

where we set $\omega_{n-1} := \operatorname{vol} \mathbb{S}_{\tilde{p}}^{n-1}$.

Lemma 2.1. *If* $\lim_{t\to\infty} \operatorname{vol} B_t(\tilde{p}) = \infty$, *then*

$$\operatorname{vol}_{\mathbb{S}_p^{n-1}} A_p \geqslant \omega_{n-1} \lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t(\tilde{p})}.$$

Proof. Let $U(A_p)$ denote any open neighborhood of A_p in $\mathbb{S}_p^{n-1}$. Since M is complete and non-compact, there exists $t_0 > 0$ such that, for any minimal geodesic segment $\gamma|_{[0,t_0]}$ emanating from p, $\gamma'(0) \in U(A_p)$. It follows from the Bishop volume comparison theorem and (2.1) that, for any $t > t_0$,

$$\operatorname{vol} B_t(p) \leqslant \int_0^{t_0} dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r, v)\, d\mathbb{S}_p^{n-1} + \int_{t_0}^t dr \int_{U(A_p)} \overline{\Theta}(r, v)\, d\mathbb{S}_p^{n-1}$$

$$\leqslant \int_0^{t_0} dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r,v)\, d\mathbb{S}_p^{n-1} + \int_{t_0}^{t} dr \int_{U(A_p)} \widetilde{\Theta}(r,v)\, d\mathbb{S}_p^{n-1}$$

$$= \int_0^{t_0} dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r,v)\, d\mathbb{S}_p^{n-1} + \mathrm{vol}_{\mathbb{S}_p^{n-1}} U(A_p) \int_{t_0}^{t} f(r)^{n-1}\, dr$$

$$= \int_0^{t_0} dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r,v)\, d\mathbb{S}_p^{n-1} + \frac{\mathrm{vol}_{\mathbb{S}_p^{n-1}} U(A_p)}{\omega_{n-1}} \big(\mathrm{vol}\, B_t(\tilde{p}) - \mathrm{vol}\, B_{t_0}(\tilde{p})\big). \tag{2.2}$$

Then, by Eq. (2.2), we have

$$\frac{\mathrm{vol}\, B_t(p)}{\mathrm{vol}\, B_t(\tilde{p})} \leqslant \frac{\int_0^{t_0} dr \int_{\mathbb{S}_p^{n-1}} \overline{\Theta}(r,v)\, d\mathbb{S}_p^{n-1}}{\mathrm{vol}\, B_t(\tilde{p})} + \frac{\mathrm{vol}_{\mathbb{S}_p^{n-1}} U(A_p)}{\omega_{n-1}} \left(1 - \frac{\mathrm{vol}\, B_{t_0}(\tilde{p})}{\mathrm{vol}\, B_t(\tilde{p})}\right).$$

Thus, we see

$$\lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(p)}{\mathrm{vol}\, B_t(\tilde{p})} \leqslant \frac{\mathrm{vol}_{\mathbb{S}_p^{n-1}} U(A_p)}{\omega_{n-1}},$$

i.e.,

$$\mathrm{vol}_{\mathbb{S}_p^{n-1}} U(A_p) \geqslant \omega_{n-1} \lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(p)}{\mathrm{vol}\, B_t(\tilde{p})}.$$

Since $U(A_p)$ is arbitrary, we hence get

$$\mathrm{vol}_{\mathbb{S}_p^{n-1}} A_p \geqslant \omega_{n-1} \lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(p)}{\mathrm{vol}\, B_t(\tilde{p})}. \qquad \square$$

Let F denote the net function for $\mathbb{S}_p^{n-1}$ on $[0,\pi]$ (see (1.3) for its definition).

Lemma 2.2. *Let $\mathbb{B}_\delta(v) \subset \mathbb{S}_p^{n-1}$ denote the open ball centered at $v \in \mathbb{S}_p^{n-1}$ with radius $\delta \in [0,\pi]$. Then, $\mathrm{vol}\, \mathbb{B}_\delta(v) = \omega_{n-1} F(\delta)$ for all $\delta \in [0,\pi]$.*

Proof. This is clear, since $\mathrm{vol}\, \mathbb{B}_\delta(v) = \omega_{n-2} \int_0^\delta \sin^{n-2} t\, dt$ holds for all $\delta \in [0,\pi]$. $\square$

Lemma 2.3. *Let δ be a constant number in $[0,\pi]$. If $\lim_{t\to\infty} \mathrm{vol}\, B_t(\tilde{p}) = \infty$ and*

$$\lim_{t\to\infty} \frac{\mathrm{vol}\, B_t(p)}{\mathrm{vol}\, B_t(\tilde{p})} \geqslant 1 - F(\delta), \tag{2.3}$$

then $\mathrm{vol}_{\mathbb{S}_p^{n-1}} A_p \geqslant \mathrm{vol}\, \mathbb{B}_{\pi-\delta}(v)$ holds for all $v \in \mathbb{S}_p^{n-1}$.

Proof. By Lemma 2.1, Lemma 2.2, and (2.3), $\mathrm{vol}_{\mathbb{S}_p^{n-1}} A_p \geqslant \omega_{n-1} - \mathrm{vol}\, \mathbb{B}_\delta(v)$ holds for all $v \in \mathbb{S}_p^{n-1}$. Hence, we get the assertion. $\square$

3. Proofs of diffeomorphism theorems

The purpose of this section is to prove Main Theorem (Theorems 3.1 and 3.4) and its corollary (Corollary 3.5). Throughout this section, let M denote a complete non-compact connected Riemannian n-dimensional manifold.

Theorem 3.1. *For any fixed point $p \in M$, there exist locally Lipschitz functions $G(t)$ (respectively $K(t)$) on $[0,\infty)$ such that radial Ricci (respectively sectional) curvature of (M,p) at p is bounded from below by that of an n-model with G (respectively that of a non-compact model surface of revolution with K) as its radial curvature function.*

Proof. We will state the outline of the proof, since the proof is the very same as that of [8, Lemma 5.1]. Let $\gamma_v : [0, \rho(v)] \to M$ denote a minimal geodesic emanating from $p = \gamma_v(0)$ such that $v = \gamma_v'(0) \in \mathbb{S}_p^{n-1}$, where $\rho(v) := \sup\{t > 0 \mid d(p, \gamma_v(t)) = t\}$. For each $v \in \mathbb{S}_p^{n-1}$, let $\mathrm{Ric}_p(\gamma_v'(t))$ be the radial Ricci curvature of M at p along γ_v. Now, we define a function G on $[0, \infty)$ by $G(t) := \min\{\mathrm{Ric}_p(\gamma_v'(\rho_t(v))) \mid v \in \mathbb{S}_p^{n-1}\}$ where $\rho_t(v) := \min\{\rho(v), t\}$. It is easy to check that $G(t)$ has the required properties.

For a locally Lipschitz function $K(t)$ on $[0, \infty)$ which bounds the radial sectional curvature of M at p from below, see [8, Lemma 5.1]. $\square$

By Theorem 3.1, we may apply a new type of the Toponogov comparison theorem to the pair (M, p) in Theorem 3.1, which was established by the present authors as generalization of the comparison theorem in conventional comparison geometry:

A new type of Toponogov Comparison Theorem. *(See [8, Theorem 4.12].) Let (X, o) be a complete non-compact Riemannian manifold X whose radial sectional curvature at the base point o is bounded from below by that of a non-compact model surface of revolution $(\widetilde{X}, \tilde{o})$ with its metric $dt^2 + h(t)^2\, d\theta^2$, $(t, \theta) \in (0, \infty) \times \mathbb{S}^1_{\tilde{o}}$. If $(\widetilde{X}, \tilde{o})$ admits a sector*

$$\widetilde{V}(\delta_0) := \left\{\tilde{x} \in \widetilde{X} \;\middle|\; 0 < \theta(\tilde{x}) < \delta_0\right\}, \quad \delta_0 \in (0, \pi],$$

having no pair of cut points, then, for every geodesic triangle $\triangle(oxy)$ in (X, o) with $\angle(xoy) < \delta_0$, there exists a geodesic triangle $\widetilde{\triangle}(oxy) := \triangle(\tilde{o}\tilde{x}\tilde{y})$ in $\widetilde{V}(\delta_0)$ such that

$$d(\tilde{o}, \tilde{x}) = d(o, x), \qquad d(\tilde{o}, \tilde{y}) = d(o, y), \qquad d(\tilde{x}, \tilde{y}) = d(x, y) \tag{3.1}$$

and that

$$\angle(xoy) \geqslant \angle(\tilde{x}\tilde{o}\tilde{y}), \qquad \angle(oxy) \geqslant \angle(\tilde{o}\tilde{x}\tilde{y}), \qquad \angle(oyx) \geqslant \angle(\tilde{o}\tilde{y}\tilde{x}).$$

Here $\angle(oxy)$ denotes the angle between the minimal geodesic segments from x to o and y forming the triangle $\triangle(oxy)$.

Notice that the assumption on $\widetilde{V}(\delta_0)$ in our comparison theorem is automatically satisfied, if we employ a von Mangoldt surface of revolution (which is, by definition, its radial curvature function is non-increasing on $[0, \infty)$), or a Cartan–Hadamard surface of revolution (which is, by definition, its radial curvature function is non-positive on $[0, \infty)$) as a $(\widetilde{X}, \tilde{o})$ for $\delta_0 \leqslant \pi$.

Remark 3.2. In [7], the present authors very recently generalized, from the radial curvature geometry's standpoint, the Toponogov comparison theorem to a complete Riemannian manifold with smooth convex boundary.

By the same argument in the proof of [8, Theorem 5.3], we have

Lemma 3.3. *(See [8, Theorem 5.3].) Let (M^*, p^*) be a non-compact model surface of revolution with its metric $dt^2 + m(t)^2\, d\theta^2$, $(t, \theta) \in (0, \infty) \times \mathbb{S}^1_{p^*}$, satisfying the differential equation $m''(t) + K(t)m(t) = 0$ with $m(0) = 0$ and $m'(0) = 1$. Here $K : [0, \infty) \to \mathbb{R}$ denotes a continuous function. If M^* satisfies*

$$\int_0^\infty tK(t)\, dt > -\infty$$

and $K(t) \leqslant 0$ on $[0, \infty)$, then

$$1 \leqslant \lim_{t\to\infty} m'(t) \leqslant \exp\left(\int_0^\infty \bigl(-tK(t)\bigr)\, dt\right) < \infty$$

holds. In particular, M^* admits a finite total curvature.

Take any $p \in M$, and fix it. From now on, for the p, let G, K be locally Lipschitz functions on $[0, \infty)$ in Theorem 3.1, respectively. Let $(\widetilde{M}^n, \tilde{p})$ denote an n-model with the G as its radial curvature function, i.e.,

$$\mathrm{Ric}_p\bigl(\gamma_v'(t)\bigr) \geqslant G\bigl(\tilde{\gamma}(t)\bigr)$$

on $[0, \infty)$, and let $B_t(p)$ (respectively $B_t(\tilde{p})$) denote the open distance ball at p with radius $t > 0$ in M (respectively the open distance ball at $\tilde{p} \in \widetilde{M}^n$ with radius $t > 0$ in $\widetilde{M}^n$). Moreover, we denote by (M^*, p^*) a non-compact model surface of revolution with its metric $g^* = dt^2 + m(t)^2\, d\theta^2$, $(t, \theta) \in (0, \infty) \times \mathbb{S}^1_{p^*}$, satisfying the differential equation

$$m''(t) + K^*(t)m(t) = 0$$

with $m(0) = 0$ and $m'(0) = 1$, where $K^* := \min\{0, G, K\}$. Notice that we may take (M^*, p^*) a comparison surface for the pair (M, p) whenever we apply a new type of the Toponogov comparison theorem to (M, p), since $K(t) \geqslant K^*(t)$ and $K^*(t) \leqslant 0$ on $[0, \infty)$.

Theorem 3.4. *If* $\lim_{t\to\infty} \operatorname{vol} B_t(\tilde{p}) = \infty$ *and*

$$\lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t(\tilde{p})} \geqslant 1 - F\big(\delta\big(K^*\big)\big) \tag{3.2}$$

then M *is diffeomorphic to Euclidean* n*-space* $\mathbb{R}^n$*. Here* F *denotes the net function for* $\mathbb{S}_p^{n-1}$*, and we set*

$$\delta\big(K^*\big) := \frac{\pi}{2} \exp\left(\int_0^\infty tK^*(t)\,dt\right).$$

Proof. We first consider the case where

$$\int_0^\infty tK^*(t)\,dt = -\infty.$$

Then, since

$$\lim_{t\to\infty} \frac{\operatorname{vol} B_t(p)}{\operatorname{vol} B_t(\tilde{p})} = 1$$

holds, M is isometric to $\widetilde{M}^n$. Hence, M is diffeomorphic to $\mathbb{R}^n$.

Next, we consider the case where

$$\int_0^\infty tK^*(t)\,dt > -\infty. \tag{3.3}$$

From the critical point theory (cf. [5], Corollary 1.4 in [10, Chapter 11]), it is sufficient to prove that any point distinct from p is not critical of $d(p, \cdot)$. Suppose that there exists a critical point $x \in M \setminus \{p\}$ of $d(p, \cdot)$. Let $\gamma : [0, d(p, x)] \to M$ be any minimal geodesic segment joining from $p = \gamma(0)$ to $x = \gamma(d(p, x))$, and let $\mu : [0, \infty) \to M$ be any ray emanating from $p = \mu(0)$. By the Cohn-Vossen's technique (see [3], or [12, Lemma 2.2.1]), there exist a divergent sequence $\{t_i\}$ and a sequence of minimal geodesic segments $\eta_i : [0, \ell_i] \to M$ emanating from $x = \eta_i(0)$ to $\mu(t_i) = \eta_i(\ell_i)$, where $\ell_i := d(x, \mu(t_i))$, such that

$$\lim_{i\to\infty} \angle\big(\eta_i'(\ell_i), \mu'(t_i)\big) = 0. \tag{3.4}$$

Since x is a critical point of $d(p, \cdot)$, for each η_i, there exists a minimal geodesic segment $\sigma_i : [0, d(p, x)] \to M$ emanating from x to p such that

$$\angle\big(\sigma_i'(0), \eta_i'(0)\big) \leqslant \pi/2. \tag{3.5}$$

Then, it follows from a new type of the Toponogov comparison theorem that there exists a geodesic triangle $\triangle(p^*x^*\mu(t_i)^*) \subset M^*$ corresponding to the triangle $\triangle(px\mu(t_i)) \subset M$ which consists of the sides γ, η_i, and $\mu|_{[0,t_i]}$ such that (3.1) holds (for $o = p$ and $y = \mu(t_i)$) and that

$$\angle\big(x^*p^*\mu(t_i)^*\big) \leqslant \angle\big(\gamma'(0), \mu'(0)\big), \tag{3.6}$$

$$\angle\big(p^*\mu(t_i)^*x^*\big) \leqslant \angle\big(p\mu(t_i)x\big). \tag{3.7}$$

By (3.4) and (3.7),

$$\lim_{i\to\infty} \angle\big(p^*\mu(t_i)^*x^*\big) = 0. \tag{3.8}$$

On the other hand, we denote by $\triangle(p\sigma_i(0)\mu(t_i)) \subset M$ the geodesic triangle consisting of the sides σ_i, η_i, and $\mu|_{[0,t_i]}$. By our Toponogov comparison theorem and (3.5), we have

$$\angle\big(p^*x^*\mu(t_i)^*\big) \leqslant \pi/2. \tag{3.9}$$

Applying the Gauss–Bonnet theorem to the geodesic triangle $\triangle(p^*x^*\mu(t_i)^*)$, we have

$$\begin{aligned}
\angle\bigl(x^*p^*\mu(t_i)^*\bigr)+\angle\bigl(p^*x^*\mu(t_i)^*\bigr)+\angle\bigl(p^*\mu(t_i)^*x^*\bigr)-\pi &= \int\limits_{\triangle(p^*x^*\mu(t_i)^*)} K^*\circ t\,dM^* \\
&\geqslant \frac{\angle(x^*p^*\mu(t_i)^*)}{2\pi}\int\limits_{M^*} K^*\circ t\,dM^* \\
&= \frac{\angle(x^*p^*\mu(t_i)^*)}{2\pi}c\bigl(M^*\bigr). \qquad (3.10)
\end{aligned}$$

Moreover, by (3.9), we have

$$\angle\bigl(x^*p^*\mu(t_i)^*\bigr)+\angle\bigl(p^*\mu(t_i)^*x^*\bigr)-\pi/2 \geqslant \angle\bigl(x^*p^*\mu(t_i)^*\bigr)+\angle\bigl(p^*x^*\mu(t_i)^*\bigr)+\angle\bigl(p^*\mu(t_i)^*x^*\bigr)-\pi \qquad (3.11)$$

Combining (3.10) and (3.11), we see

$$\angle\bigl(x^*p^*\mu(t_i)^*\bigr) \geqslant \frac{\pi(\pi-2\angle(p^*\mu(t_i)^*x^*))}{2\pi-c(M^*)}. \qquad (3.12)$$

Since $K^*(t)\leqslant 0$ on $[0,\infty)$ and (3.3), it follows from Lemma 3.3 that

$$1\leqslant \lim_{t\to\infty} m'(t)\leqslant \exp\left(\int\limits_0^\infty\bigl(-tK^*(t)\bigr)\,dt\right)<\infty.$$

Thus, by the isoperimetric inequality (cf. [12, Theorem 5.2.1]), we have

$$2\pi-c\bigl(M^*\bigr)=2\pi\lim_{t\to\infty} m'(t)\leqslant 2\pi\exp\left(\int\limits_0^\infty\bigl(-tK^*(t)\bigr)\,dt\right)<\infty. \qquad (3.13)$$

Combining (3.6), (3.12), and (3.13), we have

$$\angle\bigl(\gamma'(0),\mu'(0)\bigr)\geqslant\left(\frac{\pi}{2}-\angle\bigl(p^*\mu(t_i)^*x^*\bigr)\right)\exp\left(\int\limits_0^\infty tK^*(t)\,dt\right). \qquad (3.14)$$

Since (3.8) holds, we obtain, by taking the limit of i,

$$\angle\bigl(\gamma'(0),\mu'(0)\bigr)\geqslant\delta\bigl(K^*\bigr). \qquad (3.15)$$

Since μ is arbitrarily taken, (3.15) implies that

$$A_p\subset\overline{\mathbb{B}_{\pi-\delta(K^*)}\bigl(-\gamma'(0)\bigr)} \qquad (3.16)$$

for all minimal geodesic segments γ joining p to x. Here $-\gamma'(0)$ denotes the antipodal point of $\gamma'(0)$ in $\mathbb{S}_p^{n-1}$. Since x is a critical point of $d(p,\cdot)$, there exist at least two minimal geodesic segments joining p to x. Hence, it follows from (3.16) that there exists two distinct vectors $v_1,v_2\in\mathbb{S}_p^{n-1}$ such that $A_p\subset\overline{\mathbb{B}_{\pi-\delta(K^*)}(v_1)}\cap\overline{\mathbb{B}_{\pi-\delta(K^*)}(v_2)}$. In particular, $\mathrm{vol}_{\mathbb{S}_p^{n-1}}A_p<\mathrm{vol}\,\mathbb{B}_{\pi-\delta(K^*)}(v_1)=\mathrm{vol}\,\mathbb{B}_{\pi-\delta(K^*)}(v_2)$. This contradicts Lemma 2.3. □

Corollary 3.5. *Let (M,p) be a complete non-compact Riemannian n-manifold M whose radial sectional curvature at the base point p is bounded from below by the radial curvature function G of a non-compact model surface of revolution $(\widetilde{M},\tilde{p})$. If*

$$\lim_{t\to\infty}\frac{\mathrm{vol}\,B_t(p)}{\mathrm{vol}\,B_t^n(\tilde{p})}\geqslant 1-F\bigl(\delta(G_-)\bigr)$$

then M is diffeomorphic to Euclidean n-space $\mathbb{R}^n$. Here we denote by $B_t^n(\tilde{p})$ the open distance ball at $\tilde{p}\in\widetilde{M}^n$ with radius $t>0$ in an n-dimensional model $(\widetilde{M}^n,\tilde{p})$ of $(\widetilde{M},\tilde{p})$, and we set $G_-:=\min\{0,G\}$.

Proof. By Theorem 3.4, it is sufficient to prove the corollary in the case where

$$\lim_{t\to\infty}\mathrm{vol}\,B_t^n(\tilde{p})<\infty. \qquad (3.17)$$

Then, by (3.17)

$$\int_0^\infty f(t)^{n-1}\,dt < \infty$$

holds, where f denotes the warping function of $\widetilde{M}$. Hence, we have $\liminf_{t\to\infty} f(t) = 0$. Therefore, it follows from [11, Theorem 1.2] that M is diffeomorphic to $\mathbb{R}^n$. $\square$

Acknowledgements

The first named author would like to express to Professors J. Dodziuk, C. Sormani, N. Katz, and D. Lee his deepest gratitude for their helpful comments on the first version of our Main Theorem in the Differential Geometry Seminar at the CUNY Graduate Center, New York City, 8th and 15th September, 2009.

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